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Multiplicative Partial Integration and the Trotter Product Formula

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For a long time it has been apparent that the Trotter Product Formula, a simple version of which reads $\exp\{(A+B)t\} = \lim_{n\to\infty} \exp\{(At/n)\cdot(Bt/n)\}^n$, where A, B are (non-commuting) operators and t is real, is related to the formula for the multiplicative integral of the sum of two functions. But the precise connection between the two has not been exposed in the literature. Our purpose is to fill this lacuna. The theory of the multiplicative integral is outlined in Section 1, and the formula for the integral of the sum of two functions and the Trotter Product Formula are discussed in Section 2. Possible extensions are alluded to in Section 3.

1. INTRODUCTION

Let $\mathbb{A}$ be a Banach algebra with unit $\boldsymbol{I}$ over the field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) and normed (or renormed) so that $|\boldsymbol{I}| = 1$. Let $\mathbb{A}_{\text{inv}}$ be the multiplicative subgroup of invertible elements of $\mathbb{A}$. For what follows only non-commutative $\mathbb{A}$ will be of interest.

To define *multiplicative* (or *product*) *Riemann integration* (briefly $\hat{R}$-*integration*) for functions $\mathbf{f}$ on[1] $[a, b] \subset \mathbb{R}$ to $\mathbb{A}$ we follow the pattern of ordinary (additive) Riemann integration (briefly, *R-integration*). Let Π_a^b be the set of all finite decompositions $\pi = \{\Delta_1, ..., \Delta_n\}$ of $[a, b]$ into non-overlapping closed subintervals whose union is $[a, b]$. For $\pi, \pi' \in \Pi_a^b$ we say that π' is a refinement of π ($\pi' < \pi$) iff each Δ'_j in π' is a subinterval of some Δ_k in π. Obviously $(\Pi_a^b, <)$ is a directed poset. Given a decomposition $\pi = \{\Delta_1, ..., \Delta_n\} \in \Pi_a^b$ a set $\pi^* = \{t_1, ..., t_n\}$ such that each $t_k \in \Delta_k$ is called a *valuation of* π. We shall write val (π) for the class of all valuations of π. We shall be concerned with nets in $\mathbb{A}$ of the form

$$(\mathbf{J}(\pi, \pi^*) : \pi \in \Pi_a^b \;\&\; \pi^* \in \operatorname{val}(\pi)).$$

We say that $\lim_{\pi\downarrow} \mathbf{J}(\pi, \pi^*) = \boldsymbol{l} \in \mathbb{A}$, iff $\forall\, \varepsilon > 0$, $\exists\, \pi_\varepsilon \in \Pi_a^b$ such that

$$\pi \in \Pi_a^b \;\&\; \pi < \pi_\varepsilon \Rightarrow \forall\, \pi^* \in \operatorname{val}(\pi), \qquad |\mathbf{J}(\pi, \pi^*) - \boldsymbol{l}| < \varepsilon.$$

[1] $[a, b] =_d \{t : a \leqslant x \leqslant b\}$, $(a, b] =_d \{t : a < t \leqslant b\}$, etc.

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Next let $|\varDelta|$ be the length of $\varDelta$ and $|\pi| = \max\{|\varDelta| : \varDelta \in \pi\}$. Then we say that $\lim_{|\pi|\to 0} \mathbf{J}(\pi, \pi^*) = \boldsymbol{l} \in \mathbb{A}$, iff $\forall \varepsilon > 0$, $\exists\, \delta_\varepsilon > 0$ such that

$$\pi \in \Pi_a^b \;\&\; |\pi| < \delta_\varepsilon \;\Rightarrow\; \forall \pi^* \in \mathrm{val}(\pi), |\mathbf{J}(\pi, \pi^*) - \boldsymbol{l}| < \varepsilon.$$

We now depart from the R-theory by stipulating that when we write $\pi = \{\varDelta_1, ..., \varDelta_n\} \in \Pi_a^b$ and $\varDelta_k = [x_{k-1}, x_k]$, we imply that $a = x_0 < x_1 < \cdots < x_n = b$. With this understanding, we let $\forall \mathbf{f} \in \mathbb{A}^{[a,b]}$, $\forall \pi \in \Pi_a^b$ and $\forall \pi^* \in \mathrm{val}(\pi)$,

$$\begin{aligned} \hat{J}(\mathbf{f}, \pi, \pi^*) &= \prod_{\substack{k=1 \\ \varDelta_k \in \pi \\ t_k \in \pi^*}}^{n} \{1 + \mathbf{f}(t_k)\,|\varDelta_k|\}, \\ \hat{K}(\mathbf{f}, \pi, \pi^*) &= \prod_{\substack{k=1 \\ \varDelta_k \in \pi \\ t_k \in \pi^*}}^{n} \exp\{\mathbf{f}(t_k)\,|\varDelta_k|\}. \end{aligned} \tag{1.1}$$

1.2. Definition. (a) We say that $\mathbf{f} \in \mathbb{A}^{[a,b]}$ is *$\hat{R}$-integrable on $[a, b]$* iff $\lim_{\pi\downarrow} \hat{J}(\mathbf{f}, \pi, \pi^*)$ exists. (b) When $\mathbf{f}$ is $\hat{R}$ integrable on $[a, b]$, we call this limit the *$\hat{R}$-integral of* $\mathbf{f}$ *on* $[a, b]$ and denote it by $\hat{\int}_a^b (\boldsymbol{I} + \mathbf{f}(t)\, dt)$.

The following fundamental result reveals the scope of $\hat{R}$-integrability (cf. [10, 16.3, 8.7] and [3, Theorem 1]):

1.3. Theorem. (a) *Let* $\mathbf{f} \in \mathbb{A}^{[a,b]}$. *Then the following conditions are equivalent*:

(α) $\mathbf{f}$ *is R-integrable on* $[a, b]$,

(β) $\mathbf{f}$ *is $\hat{R}$-integrable on* $[a, b]$,

(γ) $\lim_{|\pi|\to 0} \hat{J}(\mathbf{f}, \pi, \pi^*)$ *exists.*

(b) *For R-integrable* $\mathbf{f}$ *on* $[a, b]$ *to* $\mathbb{A}$ *we have*

$$\lim_{|\pi|\to 0} \hat{J}(\mathbf{f}, \pi, \pi^*) = \lim_{\pi\downarrow} \hat{J}(\mathbf{f}, \pi, \pi^*) \in \mathbb{A}_{\mathrm{inv}}.$$

(c) *If* $\mathbf{f}$ *is bounded on* $[a, b]$ *to* $\mathbb{A}$ *and continuous a.e.* (*Lebesgue*) *on* $[a, b]$, *then* $\mathbf{f}$ *is R-integrable on* $[a, b]$.

A classic example of Graves [3, p. 166] shows that functions discontinuous everywhere on $[a, b]$ to $\mathbb{A}$ can be R-integrable, and that only when $\dim \mathbb{A} < \infty$ does the converse of 1.3(c) prevail. This makes the R-theory

[2] The symbol $\hat{\int}$ is meant to be variant of P (for product), much as $\int$ is a variant of S (for sum).

non-trivial in certain aspects.[3] Another interesting result due to Schlesinger [12, pp. 42, 43] for matrix-valued functions and readily available in the present setting (cf. [10, Sect. 24.1]),[4] is the following.

1.4. THEOREM. (a) *Let* $\mathbf{f} \in \mathbb{A}^{[a,b]}$. *Then the following conditions are equivalent*:

(α) $\mathbf{f}$ *is* $\hat{R}$*-integrable,*

(β) $\lim_{\pi\downarrow} \hat{K}(\mathbf{f}, \pi, \pi^*)$ *exists.*

(b) *For R-integrable* $\mathbf{f}$ *on* $[a, b]$ *to* $\mathbb{A}$, *we have*

$$\lim_{\pi\downarrow} \hat{K}(\mathbf{f}, \pi, \pi^*) = \lim_{|\pi| \to 0} \hat{K}(\mathbf{f}, \pi, \pi^*) = \widehat{\int_a^b} (\mathbf{I} + \mathbf{f}(t)\, dt).$$

A natural notation for the last two limits is $\widehat{\int_a^b} \exp\{\mathbf{f}(t)\, dt\}$. This exponental notation is more convenient for our present purposes than the former.[5] In terms of it we can sum up some of the above-stated results in the assertion that for all R-integrable $\mathbf{f}$ on $[a, b]$ to $\mathbb{A}$,

$$\widehat{\int_a^b} (\mathbf{I} + \mathbf{f}(t)\, dt) = \widehat{\int_a^b} \exp\{\mathbf{f}(t)\, dt\} \in \mathbb{A}_{\text{inv}}. \tag{1.5}$$

Although results in [10] are not exponentially formulated, we shall, in the light of (1.5), appeal to [10] for the corresponding exponential formulations. An easy corollary of our previous results asserts the degeneracy and redundancy of the $\hat{R}$-integral in commutative situations (cf. [10, 24.3]):

1.6. COROLLARY. *Let* $\mathbf{f}$ *be R-integrable on* $[a, b]$ *to* $\mathbb{A}$ *and let the values of* $\mathbf{f}$ *commute. Then*

$$\widehat{\int_a^b} \exp\{\mathbf{f}(t)\, dt\} = \exp\left\{\int_a^b \mathbf{f}(t)\, dt\right\}.$$

These results show how $\hat{R}$-integration imitates the exponential mapping in turning functions from $\mathbb{R}$ to the algebra $\mathbb{A}$ into functions from $\mathbb{R}$ to the group $\mathbb{A}_{\text{inv}}$. Its uses in solving time-dependent evolution equations in the uniform operator-topology are well known.

[3] The range of an R-integrable function $\mathbf{f}$ need not be separable, nor need $|\mathbf{f}(\cdot)|$ be R-integrable. Thus the R-integral is not a Bochner–Lebesgue integral. To the best of our knowledge it is not yet known if the product of two R-integrable functions is R-integrable. Example 20.2 in [10] is wrong.

[4] It fails for $\hat{R}$-Stieltjes integration in cases where the measure $|\Delta|$ is not absolutely continuous with respect to the Lebesgue measure.

[5] The exponential $\hat{K}$-formulation is the only convenient one for $\hat{R}$-Stieltjes integration, and, unlike the $\hat{I}$-formulation, continues to make sense when $\mathbb{A}$ is replaced by the Lie algebra of a Lie group, cf. [4].

2. Partial Integration and the Trotter Formula

The theorem on partial $\hat{R}$-integration [10, 29.6] reads:

2.1. Theorem. *If* f *and* g *are R-integrable on* $[a, b]$ *to* $\mathbb{A}$, *then so is the function whose value at t is*

$$\tilde{\int}_a^t \exp\{\mathbf{g}(s)ds\} \cdot \mathbf{f}(t) \cdot \left[\tilde{\int}_a^t \exp\{\mathbf{g}(s)\}\, ds\right]^{-1}, \qquad a \leqslant t \leqslant b,$$

and

$$\tilde{\int}_a^b \exp\{\mathbf{f}(t) + \mathbf{g}(t)\}\, dt$$

$$= \tilde{\int}_a^b \exp\left[\left\}\tilde{\int}_a^t \exp \mathbf{g}(s)ds\right\} \mathbf{f}(t) \left\{\tilde{\int}_a^t \exp \mathbf{g}(s)ds\right\{^{-1} dt\right] \cdot \tilde{\int}_a^b \exp\{\mathbf{g}(t)\, dt\}.$$

This is the multiplicative analogue of the additive partial integration formula

$$\int_a^b \mathbf{f}(t)\, dt \cdot \int_a^b \mathbf{g}(t)\, dt = \int_a^b \left\{\int_a^t \mathbf{f}(s)ds\right\} \mathbf{g}(t)dt + \int_a^b \mathbf{f}(t) \left\{\int_a^t \mathbf{g}(s)ds\right\} dt.$$

The roles of addition and multiplication get curiously interchanged in the two results.

On the other hand, the Trotter product theorem reads (cf. Trotter [14] and Chernoff [2]):

2.2. Theorem. *Let* (i) $\mathscr{X}$ *be a Banach space over* $\mathbb{F}$ *and* (ii) A, B *be the infinitesimal generators of two strongly continuous semigroups on* $\mathbb{R}_{0+}$ *of contraction operators on* $\mathscr{X}$ *to* $\mathscr{X}$ *of class* C_0 (*cf.* [7, p. 321]) *and such that closure* $\overline{A + B}$ *of* $A + B$ *is single-valued and is itself such a generator. Then*

$$\forall\, t \geqslant 0, \qquad \exp\{(\overline{A + B})t\} = \operatorname*{slim}_{n\to\infty} [\exp(At/n) \cdot \exp(Bt/n)]^n,$$

where slim *refers to the limit in the strong operator topology.*

When we learnt of Trotter's formula, we realized, as others acquainted with both results must have, that for *continuous* linear operators A, B, it had to be a very special instance of the multiplicative partial integration formula. The exact relationship became clear during the late 1960s, when Dr. John Hamilton began on a dissertation, in which this and other issues were discussed. This dissertation [4], unfortunately, remains unpublished. The writer is grateful to Professor Gian-Carlo Rota for the suggestion that a clear statement of the relationship between the two results would be useful to readers.

In a nutshell, Trotter's theorem (Theorem 2.2), by dint of its admittance of the strong operator topology and of discontinuous linear operators A and B, is much deeper than the Partial Integration Theorem (Theorem 2.1). But with the (rather severe) restriction that A and B be continuous on $\mathscr{X}$, it turns into a very trivial instance of Theorem 2.1. To demonstrate the last fact we shall now show that the general formula in Theorem 2.1 can be given a formulation which is closer to Trotter's:

2.3. COROLLARY. *Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be R-integrable on* $[a, b]$ *to* $\mathbb{A}$. *Then*

$$\overset{\curvearrowright}{\int_a^b} \exp\{\mathbf{f}(t) + \mathbf{g}(t)\}\, dt = \lim_{|\pi| \to 0} \prod_{\substack{k=1 \\ \Delta_k \in \pi \\ t_k \in \pi^*}}^{n} [\exp\{\mathbf{f}(t_k)\, |\Delta_k|\} \cdot \exp\{\mathbf{g}(t_k)\, |\Delta_k|\}].$$

Proof. The proof hinges on the simple identity, $\forall\, \mathbf{x}_k, \mathbf{y}_k \in \mathbb{A}$,

$$\prod_{k=1}^{n} \exp\left[\left(\prod_{j=1}^{k} \exp \mathbf{x}_j\right) \mathbf{y}_k \left(\prod_{j=1}^{k} \exp \mathbf{x}_j\right)^{-1}\right] \cdot \prod_{k=1}^{n} \exp \mathbf{x}_k$$

$$= \prod_{k=1}^{n} [(\exp \mathbf{x}_k) \cdot (\exp \mathbf{y}_k)], \tag{1}$$

which is the exponential version of the identity $[10, 29.5]$[6] which we needed to prove Theorem 2.1. To prove (1), just apply the triviality

$$\forall\, \mathbf{a} \in \mathbb{A}_{\mathrm{inv}}\ \&\ \forall\, \mathbf{b} \in \mathbb{A}, \qquad \exp(aba^{-1}) = \mathbf{a}(\exp \mathbf{b})\mathbf{a}^{-1}, \tag{2}$$

taking $\mathbf{a} = \Pi_1^k(\exp \mathbf{x}_j)$, $\mathbf{b} = \mathbf{y}_k$. This reduces the LHS of (1) to

$$\prod_{k=1}^{n} \left[\left(\prod_{j=1}^{k} \exp \mathbf{x}_j\right) (\exp \mathbf{y}_k) \left(\prod_{j=1}^{k} \exp \mathbf{x}_j\right)^{-1}\right] \cdot \prod_{k=1}^{n} \exp \mathbf{x}_k,$$

[6] To wit, $\prod_{k=1}^n [1 + \{\prod_1^k (1+y_i)\} x_k \{\prod_1^k (1+y_i)\}^{-1}] \cdot \prod_{k=1}^n (1+y_k) = \prod_{k=1}^n \{(1+x_k)(1+y_k)\}$. Incidentally, this gives the product formula $\exp\{(A+B)t\} = \lim_{n\to\infty}\{(1+A/n)(1+B/n)\}^n$ for $A, B \in \mathbb{A}$, which does not extend to the discontinuous case.

which in turn is easily seen to reduce to the RHS of (1), since the $\exp \mathbf{x}_j$ products telescope.

Now apply (1) taking $\mathbf{x}_k = \mathbf{f}(t_k)\,|\Delta_k|$, $\mathbf{y}_k = \mathbf{g}(t_k)\,|\Delta_k|$, where $\{\Delta_1,..., \Delta_n\} = \pi \in \Pi_a^b$ and $(t_1,..., t_n) = \pi^* \in \mathrm{val}(\pi)$. Then Theorem 2.1 shows at once that

$$\lim_{|\pi|\to 0} \mathrm{LHS}(1) = \int_a^b \exp\{\mathbf{g}(t) + \mathbf{f}(t)\}\,dt = \int_a^b \exp\{\mathbf{f}(t) + \mathbf{g}(t)\}\,dt.$$

Since, on the other hand,

$$\lim_{|\pi|\to 0} \mathrm{RHS}(1) = \lim_{|\pi|\to 0} \prod_{k=1}^{n} [\exp\{\mathbf{f}(t_k)\,|\Delta_k| \cdot \exp\{\mathbf{g}(t_k\,|\Delta_k|\,|,$$

we are done. ∎

When $[a, b] = [0, T]$, $T > 0$, and $\pi \in \Pi_0^T$ is the uniform partition with n cells $\Delta_k = [(k-1)T/n,\ kT/n]$, so that each $|\Delta_k| = T/n$, and $\pi^* = \{T/n\ 2T/n,..., T\}$, i.e., π^* is the "right-most" valuation of π, the formula in Corollary 2.3 becomes

$$\int_0^T \exp\{\mathbf{f}(t) + \mathbf{g}(t)\}\,dt$$
$$= \lim_{n\to\infty} \prod_{k=1}^{n} [\exp\{\mathbf{f}(kT/n)T/n\} \cdot \exp\{\mathbf{g}(kT/n)T/n\}], \tag{2.4}$$

which is even closer to Trotter's formula in Theorem 2.2. In the very trivial case

$$\mathbf{f}(t) = A, \qquad \mathbf{g}(t) = B, \qquad a \leqslant t \leqslant b,$$

(2.4) reduces to the latter formula. To see this we must note that the values of $\mathbf{f} + \mathbf{g}$ now commute and that Corollary 1.6 applies. Here $\mathbb{A}$ is of course taken to be the Banach algebra of continuous linear operators on $\mathscr{X}$ to $\mathscr{X}$, and it is assumed that A, B are in $\mathbb{A}$.

The formula in Theorem 2.1 admits another reformulation which may be useful. Let $\mathbf{f}$ be R-integrable on $[a, b]$ to $\mathbb{A}$,

$$\wp \underset{\mathrm{d}}{=} \{(c, d]: a \leqslant c \leqslant d \leqslant b\},$$

and let

$$\forall\,(c, d] \in \wp, \qquad \hat{M}_{\mathbf{f}}(c, d] \underset{\mathrm{d}}{=} \int_c^d \exp\{\mathbf{f}(t)\,dt\} \in \mathbb{A}_{\mathrm{inv}}. \tag{2.5}$$

It is known (cf. [10, 12.4–12.6]) that the measure $\hat{M}_{\mathbf{f}}$ on the pre-ring $\wp$ to $\mathbb{A}_{\mathrm{inv}}$ is well defined, and is *finitely multiplicative* in the sense that for disjoint $J_1, J_2 \in \wp$ for which $J_1 \cup J_2 \in \wp$ and J_1 *precedes* J_2, we have

$$\hat{M}_{\mathbf{f}}(J_1 \cup J_2) = \hat{M}_{\mathbf{f}}(J_1) \cdot \hat{M}_{\mathbf{f}}(J_2).$$

We now assert the following corollary:

2.6. COROLLARY. *Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be R-integrable on* $[a, b]$ *to* $\mathbb{A}$. *Then (cf.* (2.5)),

$$\prod_a^b \exp\{\mathbf{f}(t) + \mathbf{g}(t)\}\, dt = \lim_{|\pi| \to 0} \prod_{\substack{k=1 \\ \Delta_k \in \pi \\ t_k \in \pi^*}}^{n} [\hat{M}_{\mathbf{g}}(\Delta_k') \cdot \exp\{\mathbf{f}(t_k)\, |\Delta_k|\}],$$

where Δ_k' *is the interval resulting from* Δ_k *after removal of its left endpoint.*

Proof. By Theorem 2.1, we have (cf. (1.1))

$$\mathrm{LHS} = \lim_{|\pi| \to 0} \hat{R}(\hat{M}_{\mathbf{g}}(a, \cdot\,]\, \mathbf{f}(\,\cdot\,) \hat{M}_{\mathbf{g}}(a, \cdot\,]^{-1}, \pi, \pi^*) \cdot \hat{M}_{\mathbf{g}}(a, b].$$

Now take $\pi = \{\Delta_1, ..., \Delta_n\}$, with $\Delta_k = [x_{k-1}, x_k]$, and $\pi^* = \{x_1, ..., x_n\}$, i.e., the "right-most" valuation of π. An application of the triviality marked (2) in the proof of Corollary 2.3 then yields a product, which, because of the multiplicativity of $\hat{M}_{\mathbf{g}}$, telescopes into the desired expression. Since the argument is very close to that used in proving Corollary 2.3, we omit the details. ∎

When the values of one of the functions $\mathbf{f}$, $\mathbf{g}$ commute, we easily deduce (cf. Corollary 1.6):

2.7. COROLLARY. *Let* $\mathbf{f}, \mathbf{g}$ *be R-integrable on* $[a, b]$ *to* $\mathbb{A}$, *and let the values of* $\mathbf{g}$ *commute. Then*

$$\begin{aligned} &\prod_a^b \exp\{\mathbf{f}(t) + \mathbf{g}(t)\}\, dt \\ &\qquad = \lim_{|\pi| \to 0} \prod_{\substack{k=1 \\ \Delta_k \in \pi \\ t_k \in \pi^*}}^{n} \left[\exp\left\{\int_{\Delta_k} \mathbf{g}(s)\, ds\right\} \cdot \exp\{\mathbf{f}(t_k)\, |\Delta_k|\}\right]. \end{aligned}$$

The Trotter formula is again obtainable from this corollary by the substitution $\mathbf{f}(t) = A$, $\mathbf{g}(t) = B$, $a \leqslant t \leqslant b$.

Trotter's Theorem is valuable in the perturbation theory of strongly continuous semi-groups. It has proved especially significant in the derivation of the Feynman–Kac formula (cf. Nelson [11]). On the other hand, the multiplicative partial integration formula is central to the perturbation theory of *all* (time-dependent) evolution equations, but under the severe restriction that the operators be continuous and the topology be of uniform convergence. But even this restricted set-up is useful for certain problems in quantum mechanics (cf., e.g., Hamilton and Schulman [5] and Kramer [9]).

A question which naturally arises is whether the unrestricted Trotter formula can be reached from its restricted version (deduced above) by a limiting argument based on semi-group theory. Some attempts by this writer show that the difficulties encountered in effecting such a transition are about as severe as those met with in a direct approach, and that Chernoff's direct proof [2] is still perhaps the best available.

3. Extensions

The partial integration formula survives for multiplicative integration of functions from $\mathbb{R}$ to the Lie algebra $\mathfrak{g}$ of a Lie Group $\mathscr{G}$. In this theory, the possibility of which was surmised by G. Birkhoff [1, Sect. 4], $\mathbb{A}$ gets replaced by $\mathfrak{g}$ and $\mathbb{A}_{\text{inv}}$ by $\mathscr{G}$. The following formulation, involving the so-called adjoint representation (cf. Helgason [6, p. 90]) is due to Hamilton [4, p. 54]:

3.1. Theorem. *Let* (i) $\mathfrak{g}$ *be the Lie algebra of a Lie group* $\mathscr{G}$, (ii) f, g *be R-integrable on functions on* $[a, b]$ *to* $\mathfrak{g}$, (iii) Ad *be the adjoint representation on* $\mathscr{G}$ *to the group of invertible linear transformations on* $\mathfrak{g}$ *onto* $\mathfrak{g}$. *Then*

$$\widehat{\int_a^b} \exp\{f(t) + g(t)\}\, dt$$

$$= \widehat{\int_a^b} \exp\left[\left\{\mathrm{Ad}\left(\widehat{\int_a^t} \exp g(s)\,ds\right)\right\}\{f(t)\}\,dt\right] \cdot \widehat{\int_a^b} \exp\, g(t)\,dt.$$

The Trotter product formula for $A, B \in \mathfrak{g}$ is again a trivial instance of this formula, but as Hamilton notes [4, p. 57], it is an almost immediate consequence of the corresponding formula for $n \times n$ matrices due to Lie.

The generalization of the partial integration formula needed in order to subsume Trotter's result (Theorem 2.2) in its full generality must of course cover functions **f**, **g** whose values are certain discontinuous linear operators. Such a generalization would demand the demarcation of a suitable subclass $\mathbb{B}$ of discontinuous linear operators from a Banach space $\mathscr{X}$ to itself, and the extension of the theory outlined in Section 1 to cover functions from real intervals to this $\mathbb{B}$. The multiplicative integral of such a function should turn out to be a continuous linear operator on $\mathscr{X}$ to $\mathscr{X}$. To the best of our knowledge, a theory of this sort has not yet been developed. Kato's paper [8], though not addressed to this issue, offers some valuable insights as to how we might proceed.

REFERENCES

1. G. BIRKHOFF, On product integration, *J. Math. Phys.* **16** (1937), 105–132.
2. P. R. CHERNOFF, Note on product formulas for operator semigroups, *J. Functional Analysis* **2** (1968), 238–242.
3. L. M. GRAVES, Riemann integration and Taylor's theorem in general analysis, *Trans. Amer. Math. Soc.* **29** (1927), 163–177.
4. J. F. HAMILTON, "Multiplicative Riemann Integration and Logarithmic Differentiation in Lie Algebras and Lie Groups," Indiana University Dissertation, 1973, unpublished.
5. J. F. HAMILTON AND L. S. SCHULMAN, Path integrals and product integrals, *J. Math. Phys.* **12** (1971), 160–164.
6. S. HELGASON, "Differential Geometry and Symmetric Spaces," Academic Press, New York, 1962.
7. E. HILLE AND R. S. PHILLIPS, "Functional Analysis and Semigroups," revised ed., Amer. Math. Soc. Colloq. Pub. Vol. 31, Amer. Math. Soc., Providence, R. I., 1957.
8. T. KATO, Integration of the evolution equation in a Banach space, *J. Math. Soc. Japan* **5** (1953), 208–234.
9. I. KRAMER, The product-integral calculus formulation of quantum mechanics, *Amer. J. Phys.* **40** (1972), 1221–1230.
10. P. R. MASANI, Multiplicative Riemann integration in normed rings, *Trans. Amer. Math. Soc.* **61** (1947), 147–192.
11. E. NELSON, Feynman integrals and the Schrodinger equation, *J. Math. Phys.* **5** (1964), 332–343.
12. L. SCHLESINGER, Neue Grundlagen fur einen Infinitesimalkalkul der Matrizen, *Math. Z.* **33** (1931), 33–61.
13. H. F. TROTTER, Approximation of semigroups of operators, *Pacific J. Math.* **8** (1958), 887–919.
14. H. F. TROTTER, On the product of semigroups of operators, *Proc. Amer Math. Soc.* **10** (1959), 545–551.